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A PROOF OF AN EXTENSION OF THE ICOSAHEDRAL CONJECTURE OF STEINER FOR GENERALIZED DELTAHEDRA

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Dedicated to my father on the occasion of his 50th birthday.

Abstract. In this note we introduce a new family of convex polyhedra which we call the family of *generalized deltahedra* or in short, the family of *g-deltahedra*. Here a *g-deltahedron* is a convex polyhedron in Euclidean 3-space whose each face is an edge-to-edge union of some triangles each being congruent to a given regular triangle. Steiner's famous *icosahedral conjecture* (1841) says that among all convex polyhedra isomorphic to an icosahedron (that is having the same face structure as an icosahedron) the regular icosahedron has the smallest isoperimetric quotient. In this paper we prove that the regular icosahedron has the smallest isoperimetric quotient among all g-deltahedra.

1. Introduction

The origin of the isoperimetric inequality is lost in the beginning of the history of mathematics. The first proof of the isoperimetric property of the circle is due to Zenodorus [8], according to which among all planar domains of given area the circular disk has the least perimeter. In fact, Zenodorus proved that among all polygons of given number of sides and of given area, the regular convex one has the least possible perimeter. This property implies the isoperimetric property of the circle by a standard approximation argument.

For isoperimetric problems in Euclidean 3-space it seems useful to recall the notion of isoperimetric quotient. Let $\mathbf{K}$ be a convex body (i.e. a compact convex set with nonempty interior) in Euclidean 3-space with surface area

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$A(\mathbf{K})$ and volume $V(\mathbf{K})$. Then the *isoperimetric quotient* $\mathrm{IQ}(\mathbf{K})$ of $\mathbf{K}$ is defined by

$$\mathrm{IQ}(\mathbf{K}) = \frac{A^3(\mathbf{K})}{V^2(\mathbf{K})}.$$

Now, on the one hand, the isoperimetric property of the sphere can be phrased as follows: among all (convex) bodies of the same volume, the round ball has the least surface area. On the other hand, putting it in a somewhat different form one can state the following: If $\mathbf{K}$ is a (convex) body in Euclidean 3-space, then $\mathrm{IQ}(\mathbf{K}) \geq 36\pi = 113.0973\dots$. Without even trying to be exhaustive, the list of mathematicians that have considered the above isoperimetric problem includes Euler, Gauss, Steiner, Weierstrass, Schwarz and Schmidt (for more details see [2]).

In a highly influential paper published in 1841, Steiner [10] proved that among all convex polyhedra of Euclidean 3-space having the same face structure as a tetrahedron (resp., octahedron) the regular tetrahedron (resp., regular octahedron) has the smallest isoperimetric quotient namely, $374.1229\dots$ (resp., $187.0614\dots$). About hundred years later Fejes Tóth [5] showed that among all convex polyhedra with 6 (resp., 12) faces the cube (resp., regular dodecahedron) has the smallest isoperimetric quotient namely, 216 (resp., $149.8578\dots$). In the above quoted paper of Steiner, his famous *icosahedral conjecture* was phrased according to which among all convex polyhedra isomorphic to an icosahedron the regular icosahedron has the smallest isoperimetric quotient namely, $136.4595\dots$. In fact, Fejes Tóth [5] conjectures that the isoperimetric quotient of the regular icosahedron (resp., regular octahedron) is the smallest isoperimetric quotient among convex polyhedra with 12 (resp., 6) vertices. If indeed someone were able to prove Steiner's icosahedral conjecture, then the five Platonic solids would minimize the isoperimetric quotient among all polyhedra of the same class.

In order to describe the main goal of this paper we need one additional input. Namely, recall that a *convex deltahedron* or for the purpose of this paper in short, a *deltahedron* is a convex polyhedron in Euclidean 3-space whose faces are congruent equilateral triangles. Deltahedra have been around for quite some time (see for example [11]) in particular, Rausenberger [9] in 1915 found exactly 8 different types of them up to similarity. (See also the paper [6] of Freudenthal and van der Waerden.) It turns out that if n denotes the number of faces of a deltahedron, then $n \in \{4, 6, 8, 10, 12, 14, 16, 20\}$. Although it is natural to extend the family of deltahedra in the way it is given in the following definition, it seems to us that our definition below is a new one and therefore it leads to a new class of convex polyhedra that has not been introduced and investigated before.

Definition 1.1. *A* **generalized deltahedron***, or* **g-deltahedron***, is a convex polyhedron in Euclidean* 3*-space whose each face is an edge-to-edge union of some triangles each being congruent to a given regular triangle.*

The family of g-deltahedra is a great deal larger and more complex than the family of deltahedra. The classification problem of g-deltahedra is the central topic of the work [1] in preparation. The main goal of this paper is to give a proof of an extension of the icosahedral conjecture of Steiner for *g-deltahedra*. Namely, we prove the following theorem.

Theorem 1.2. *The regular icosahedron has the smallest isoperimetric quotient among all g-deltahedra.*

Remark 1.3. *In fact, the proof presented below shows that the smallest isoperimetric quotient of g-deltahedra is attained by the regular icosahedron only.*

2. Proof of Theorem 1.2

Our proof is based on a sequence of lemmas.

Lemma 2.1. *For an arbitrary g-deltahedron, the number of vertices (resp., faces) is at most* 12 *(resp.,* 20*).*

Proof. Let $\mathbf{P}$ be a g-deltahedron. First, observe that as each face of $\mathbf{P}$ is a convex polygon being an edge-to-edge union of some regular triangles therefore each face of $\mathbf{P}$ is either a triangle or a quadrilateral or a pentagon or a hexagon. As a result we get that any angle of a face (*i.e.* any face angle) of $\mathbf{P}$ is either of measure $\pi/3$ or $2\pi/3$.

Second, recall (see for example [4]) that the Gauss curvature of a vertex of a convex polyhedron is defined as 2π minus the sum of the radian measures of the face angles meeting at the given vertex. Then according to a well-known theorem of Gauss (see [4]) the sum of the Gauss curvatures of the vertices of an arbitrary convex polyhedron is always 4π.

All this means that on the one hand, the Gauss curvature of any vertex of $\mathbf{P}$ is at least $\pi/3$ (in fact, the Gauss curvature of any vertex of $\mathbf{P}$ is always an integer multiple of $\pi/3$) on the other hand, as $12(\pi/3) = 4\pi$ therefore the above theorem of Gauss implies that the number of vertices of $\mathbf{P}$ is indeed at most 12.

Now, let f, e and v denote the number of faces, edges and vertices of $\mathbf{P}$. We want to show that $f \leq 20$. If we introduce the notation f_i for the number of faces of $\mathbf{P}$ having exactly i edges, then we have $3 \leq i \leq 6$, and two equations namely,

$$f_3 + f_4 + f_5 + f_6 = f \quad \text{and} \quad 3f_3 + 4f_4 + 5f_5 + 6f_6 = 2e.$$

Thus, Euler's equation ([4]) yields $2f + 2v = 2e + 4 \geq 3f + 4$. Hence,

$$f \leq 2v - 4 \leq 2 \cdot 12 - 4 = 20.$$

This completes the proof of Lemma 2.1. □

The following theorem was conjectured and partly proved by Goldberg [7] in 1935. A complete proof with a highly applicable method was given by Fejes Tóth [5] in 1948.

Lemma 2.2. *Let* $\mathbf{P}$ *be a convex polyhedron with* n *faces. Then*

$$\mathrm{IQ}(\mathbf{P}) \geq 54(n-2)\tan\omega_n(4\sin^2\omega_n - 1),$$

where

$$\omega_n = \frac{\pi n}{6(n-2)},$$

with equality just in three cases namely, when $\mathbf{P}$ *is either a regular tetrahedron* ($n = 4$) *or a cube* ($n = 6$) *or a regular dodecahedron* ($n = 12$).

Corollary 2.3. *The isoperimetric quotient of any convex polyhedron with at most* 17 *faces is strictly larger than the isoperimetric quotient of a regular icosahedron.*

Proof. Let $f(n) = 54(n-2)\tan\omega_n(4\sin^2\omega_n - 1)$. As

$$\begin{array}{rclcrcll}
f(4) & = & 374.1229\ldots & > & f(5) & = & 260.1145\ldots & > \\
f(6) & = & 216 & > & f(7) & = & 192.2854\ldots & > \\
f(8) & = & 177.4494\ldots & > & f(9) & = & 167.2893\ldots & > \\
f(10) & = & 159.8958\ldots & > & f(11) & = & 154.2749\ldots & > \\
f(12) & = & 149.8578\ldots & > & f(13) & = & 146.2954\ldots & > \\
f(14) & = & 143.3618\ldots & > & f(15) & = & 140.9040\ldots & > \\
f(16) & = & 138.8151\ldots & > & f(17) & = & 137.0178\ldots & > \quad 136.4595\ldots,
\end{array}$$

the claim follows via Lemma 2.2. □

Thus, Lemma 2.1 and Corollary 2.3 imply that in order to complete the proof of Theorem 1.2 one has to look at g-deltahedra with 20 or 19 or 18 faces and show that their isoperimetric qoutients are always larger than or equal to the isoperimetric quotient of a regular icosahedron.

The following theorem is called Cauchy's rigidity theorem that was proved by Cauchy [3] in 1813 (see also [4]).

Lemma 2.4. *If* $\mathbf{P}$ *and* $\mathbf{Q}$ *are isomorphic convex polyhedra with corresponding faces being congruent and with corresponding edges being of equal length, then* $\mathbf{P}$ *and* $\mathbf{Q}$ *are congruent.*

Lemma 2.5. *If* $\mathbf{P}$ *is a g-deltahedron with* 20 *faces, then* $\mathbf{P}$ *must be a regular icosahedron.*

Proof. The proof of Lemma 2.1 easily implies that as $\mathbf{P}$ has 20 faces therefore it must have 12 vertices and 30 edges moreover, all its faces have to be triangles. As $\mathbf{P}$ is a g-deltahedron therefore its faces are regular triangles say, of edge length 1. This means also that each vertex of $\mathbf{P}$ has at most 5 edges. However, the number of edges (resp., vertices) of $\mathbf{P}$ is 30 (resp., 12) and so, each vertex of $\mathbf{P}$ has to have exactly 5 edges. As a result we get that $\mathbf{P}$ is isomorphic to the regular icosahedron $\mathbf{Q}$ of edge length 1. Thus, Lemma 2.4 applied to $\mathbf{P}$ and $\mathbf{Q}$ implies that $\mathbf{P}$ and $\mathbf{Q}$ are congruent and therefore $\mathbf{P}$ is itself a regular icosahedron finishing the proof of Lemma 2.5. □

Lemma 2.6. *There is no g-deltahedron with exactly* 19 *faces.*

Proof. The proof below is an indirect one. So, assume that $\mathbf{P}$ is a g-deltahedron with 19 faces. Let e and v denote the number of edges and vertices of $\mathbf{P}$. Then Euler's equation yields that $19 + v = e + 2$. Note that $3 \cdot 19 \leq 2e$ and therefore $29 \leq e$. The last inequality substituted in Euler's equation yields that $19+v \geq 29+2 = 31$ implying that $v \geq 12$. Now, recall that according to Lemma 2.1 $v \leq 12$ and therefore $v = 12$ and so, Euler's equation implies that $e = 29$. Next, let m denote the number of triangular faces of $\mathbf{P}$ and let n denote the number of faces of $\mathbf{P}$ with at least 4 sides. Obviously,

$$m + n = 19. \tag{1}$$

Clearly,

$$3m + 4n \leq 2 \cdot 29 = 58. \tag{2}$$

First, (1) yields that $n = 19 - m$ and therefore (2) reads as

$$3m + 4(19 - m) \leq 58,$$

that is $18 \leq m$. Second, (1) implies that $m \leq 19$. Third, note that although $m = 19, n = 0$ is a solution of (1) and (2) still $\mathbf{P}$ does not exist for those values because, then $3 \cdot 19 = 58$ should hold, a contradiction. Thus, the only possible solution left for (1) and (2) is $m = 18, n = 1$. This means that $\mathbf{P}$ has 18 regular triangle faces and 1 quadrilateral face. (Here the nontriangular face has to be a quadrilateral because for $m = 18, n = 1$ (2) holds with equality.) But, then all the regular triangle faces of $\mathbf{P}$ are of the same side length say, of side length 1 and therefore the quadrilateral face has to be a rhombus of side length 1 with angles $\pi/3, 2\pi/3$.

Now, dissect the rhombus in question into two regular triangles along its shorter diagonal. That way the surface of $\mathbf{P}$ will consist of 20 regular triangles each of edge length 1 and just as in the proof of Lemma 2.5 we get via Cauchy's rigidity theorem that $\mathbf{P}$ is a regular icosahedron of edge length 1, a contradiction. This finishes the proof of Lemma 2.6. □

Lemma 2.7. *There is no g-deltahedron with exactly* 18 *faces.*

Proof. Again the proof below is an indirect one. So, assume that $\mathbf{P}$ is a g-deltahedron with 18 faces. Let e and v denote the number of edges and vertices of $\mathbf{P}$. Then Euler's equation yields that

$$18 + v = e + 2.$$

Note that $3 \cdot 18 \leq 2e$ and therefore $27 \leq e$. The last inequality substituted in Euler's equation yields that

$$18 + v \geq 27 + 2 = 29$$

implying that $v \geq 11$. Now, recall that according to Lemma 2.1 $v \leq 12$ and therefore either $v = 11$ or $v = 12$. Next, let m denote the number of triangular faces of **P** and let n denote the number of faces of **P** with at least 4 sides. Clearly,

$$m + n = 18. \tag{3}$$

If $v = 11$, then Euler's equation implies that $e = 27$. Moreover, it is clear that

$$3m + 4n \leq 2 \cdot 27 = 54. \tag{4}$$

First, (3) yields that $n = 18 - m$ and therefore (4) reads as

$$3m + 4(18 - m) \leq 54$$

that is $18 \leq m$. Second, (3) implies that $m \leq 18$. Therefore $m = 18$ and $n = 0$. This means that **P** has 18 triangular faces each being congruent to a given regular triangle that is **P** is a deltahedron with 18 faces. However, such a convex polyhedron cannot exist as it is shown in [9], a contradiction.

If $v = 12$, then Euler's equation implies that $e = 28$. Moreover, it is clear that

$$3m + 4n \leq 2 \cdot 28 = 56. \tag{5}$$

First, (3) yields that $n = 18 - m$ and therefore (5) reads as

$$3m + 4(18 - m) \leq 56,$$

that is $16 \leq m$. Second, (3) implies that $m \leq 18$. Thus, we get that

$$16 \leq m \leq 18.$$

If $m = 18$, then (3) implies that $n = 0$ and as before this is impossible. If $m = 17$, then (3) implies that $n = 1$. Now, it is easy to see that the non-triangular face of **P** must be a rhombus (of side length equal to the side length of the 17 regular triangle faces of **P**). But then disecting the rhombus in question by its shorter diagonal into two congruent regular triangles we would end up with an edge graph of 19 triangular faces, a contradiction (because $3 \cdot 19$ should be equal to $2 \cdot 29$). Finally, if $m = 16$, then (3) implies that $n = 2$. Here it is easy to see that the two non-triangular faces of **P** must be congruent rhombi (of side length equal to the side length of the 16 regular triangle faces of **P**). Disecting both rhombi by their shorter diagonals into congruent regular triangles we get an edge-to-edge system of 20 congruent regular triangles forming the boundary of **P**. Now, using Cauchy's rigidity theorem just as in the proof of Lemma 2.5 we get the desired contradiction. This completes the proof of Lemma 2.7.

□

Thus, the proof of Theorem 1.2 is finished.

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